SOLUTION TO WAVEGUIDE PROBLEMS BY SUCCESSIVE EXTRAPOLATED RELAXATION

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Abstract

A successive extrapolated relaxation (SER) technique has been developed to solve elliptic partial difference equations iteratively. SER is more efficient than optimized successive over-relaxation and permits useful solutions of waveguide modes using finite difference methods.

Introduction

Beaubien and Wexler[1,2] have discussed the solution of waveguide problems by finite difference methods. The resulting method PDSOR (positive definite successive over-relaxation) uses a one-dimensional search technique to obtain the best value of the over-relaxation factor. Such a technique suffers from suboptimal choice of the over-relaxation factor during the final steps of the solution. They cannot use the SOOR (successive optimized over-relaxation) techniques developed by Carré and others[3,4,5] because their matrix C although positive definite does not possess Young's Property A[6].

A new method called SER (successive extrapolated relaxation)[7] has been developed to solve elliptic partial difference equations. It has been shown that SER is at least as efficient as SOOR. Since SER does not require that the system matrix possess Young's Property A, it may be applied directly to the Beaubien and Wexler formulation with a resulting increase in speed. If the problem can be reformulated so that the system matrix does possess Young's Property A, then a refinement of SER called SEOR (successive extrapolated optimized relaxation) may be used which optimizes the pertinent parameter.

The SER Method

Let us define $V_{k,\ell}^{(n)}$ to be the value of the potential at the k,ℓ lattice point after the n^{th} iteration. During the iterative solution, the sequence $V_{k,\ell}^{(n-2)}$, $V_{k,\ell}^{(n-1)}$, $V_{k,\ell}^{(n)}$ obtained by any successive relaxation process may be plotted as shown in Fig. 1. If one assumes that the approach to the asymptotic value is characterized by an exponential behaviour

$$V_{k,\ell}^{(n)} = V_{k,\ell}^{(\infty)} + (V_{k,\ell}^{(0)} - V_{k,\ell}^{(\infty)})\,\alpha^n \qquad (1)$$

then it has been shown[7] that an approximation $A_{k,\ell}^{(n)}$ to the asymptotic value is given by:

$$A_{k,\ell}^{(n)} = \frac{[V_{k,\ell}^{(n-1)}]^2 - V_{k,\ell}^{(n-2)}\,V_{k,\ell}^{(n)}}{2V_{k,\ell}^{(n-1)} - V_{k,\ell}^{(n-2)} - V_{k,\ell}^{(n)}} \qquad (2)$$

If the sequence converged geometrically then (2) would be the solution to the problem. If the convergence is quasigeometric then (2) will be much closer to the final answer than $V_{k,\ell}^{(n)}$. In general the convergence is linear and (2) has to be modified in order to assure convergence.

The first modification involves putting a bound on the extrapolation as illustrated in Fig. 2. The second modification is due to the fact that (2) extrapolates the wrong way if there is an apparent divergence in the sequence of potentials. This is solved by reflecting the n-2 point as shown in Fig. 3. The resulting extrapolation formula is given by

$$A_{k,\ell}^{(n+1)} = \begin{cases} V_{k,\ell}^{(n)} + 2(1+\gamma)\,\Delta_2 \ ; & \Delta_1\Delta_2 \geq 0 \quad |\Delta_2| \leq |\Delta_1| \\ V_{k,\ell}^{(n)} + (3+2\gamma)\,\Delta_1 \ ; & \Delta_1\Delta_2 \geq 0 \quad |\Delta_2| > |\Delta_1| \\ V_{k,\ell}^{(n)} \ ; & \Delta_1\Delta_2 < 0 \end{cases} \qquad (3)$$

where

$$\Delta_1 = V_{k,\ell}^{(n-1)} - V_{k,\ell}^{(n-2)} \qquad (4)$$

$$\Delta_2 = V_{k,\ell}^{(n)} - V_{k,\ell}^{(n-1)} \qquad (5)$$

and γ is a constant of the order of ½. In SEOR γ is optimized.

Illustrative Example

The dominant mode for the waveguide whose cross-section is shown in Fig. 4 was solved on a CDC 6400. It took 100 iterations and 12.3 seconds to obtain the solution illustrated in Fig. 5. A five point formula was used for this process. The error criterion in this case was a residual ratio of 10^{-4}.

Higher order modes require at least a thirteen point operator, such as Beaubien and Wexler's[1], in order to have a positive definite system matrix. If one uses a seventeen point operator published by Tee[8], in conjunction with the 5-point operator one has Young's Property A. In this case one can use SEOR.

Conclusion

A more efficient algorithm has been presented for the solution of finite difference problems. When applied to microwave problems speedy solutions may be obtained.

References

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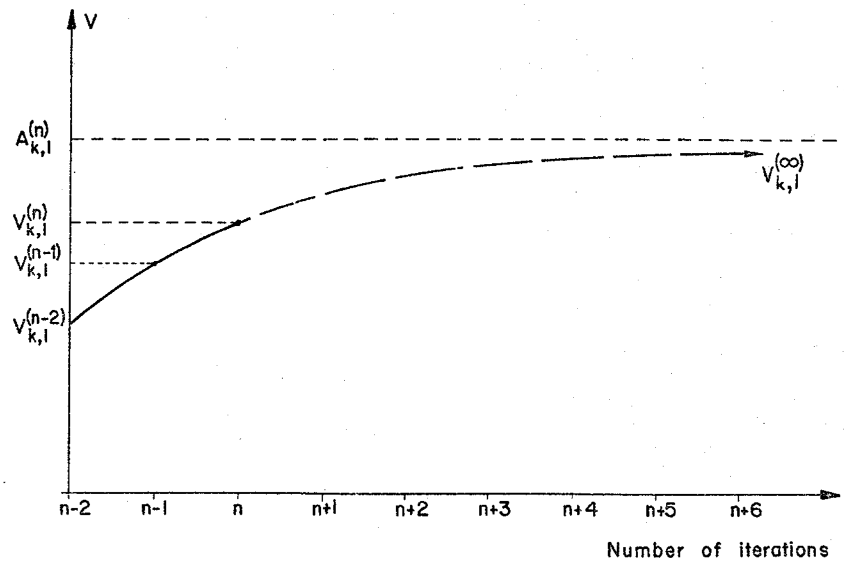

FIG. 1 THE SEQUENCE $V_{k,\ell}^{(n)}$ OBTAINED BY ANY SUCCESSIVE RELAXATION PROCESS INDICATING THE IMPLIED ASYMPTOTE.

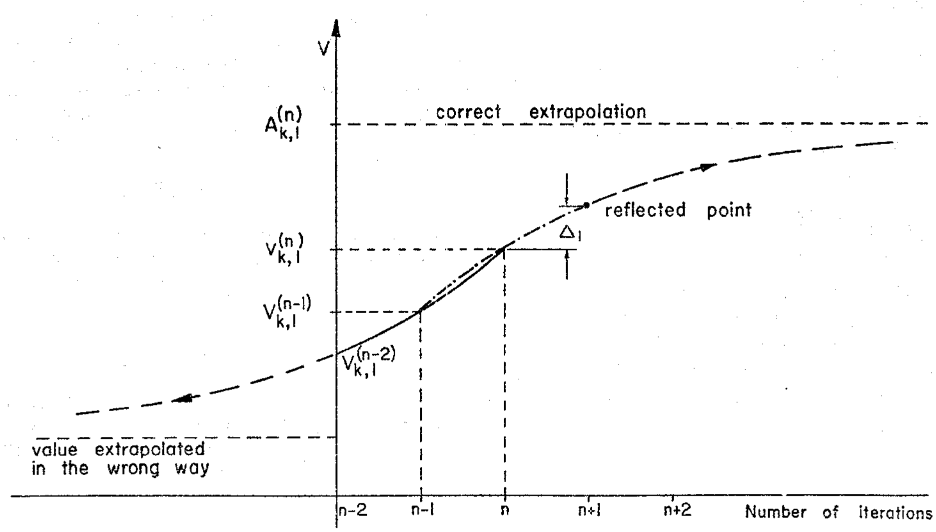

FIG. 2 BOUNDING THE EXTRAPOLATION PROCESS.

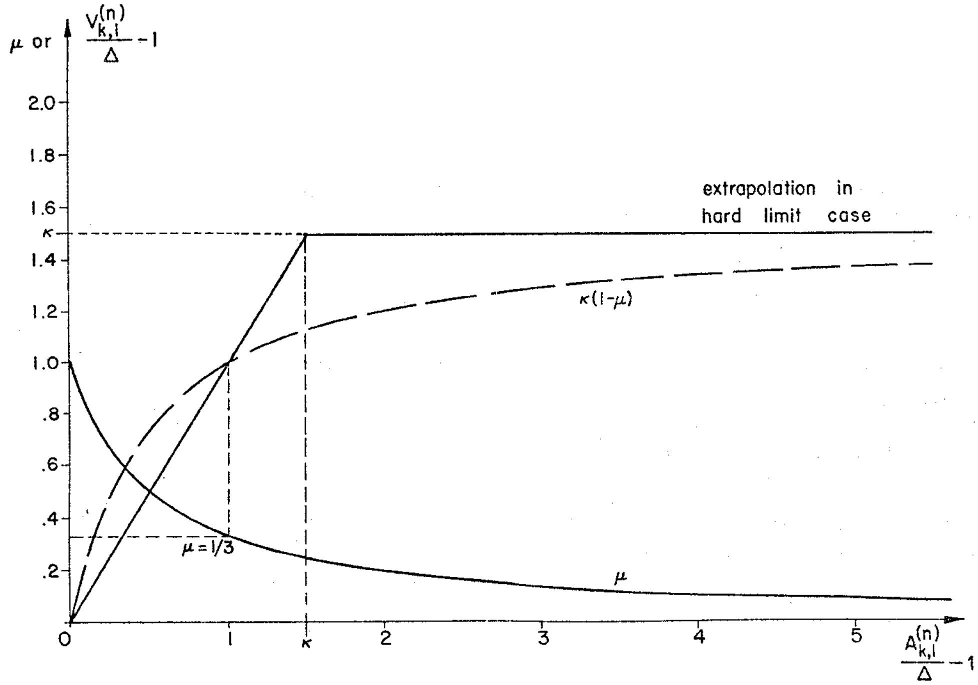

FIG. 3 REFLECTION OF THE (n-2) POINT IN ORDER TO OBTAIN CORRECT EXTRAPOLATION.

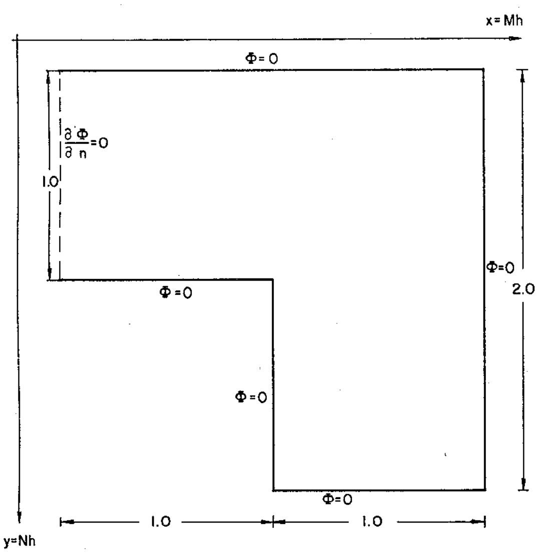

FIG. 4 BOUNDARY CONDITIONS FOR THE DOMINANT TM MODE IN A RIDGE WAVEGUIDE. ONLY ONE-HALF THE SYMMETRICAL GUIDE IS SHOWN.

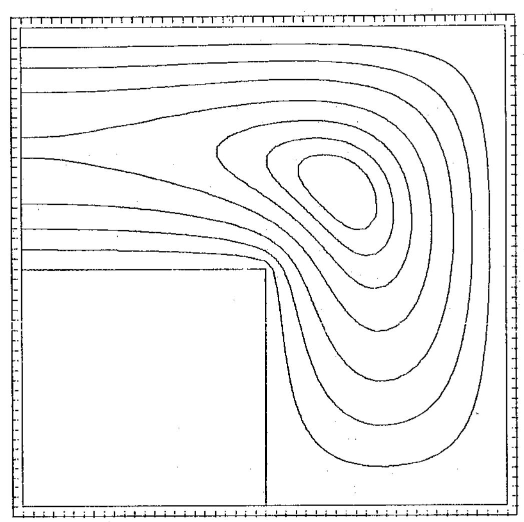

FIG. 5 THE DOMINANT TM MODE IN A SYMMETRICAL RIDGE WAVEGUIDE SHOWN IN FIGURE 4.